CHAPTER 2
BASIC VIBRATION THEORY

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INTRODUCTION

This chapter presents the theory of free and forced steady-state vibration of single degree-of-freedom systems. Undamped systems and systems having viscous damping and structural damping are included. Multiple degree-of-freedom systems are discussed, including the normal-mode theory of linear elastic structures and Lagrange's equations.

ELEMENTARY PARTS OF VIBRATORY SYSTEMS

Vibratory systems comprise means for storing potential energy (spring), means for storing kinetic energy (mass or inertia), and means by which the energy is gradually lost (damper). The vibration of a system involves the alternating transfer of energy between its potential and kinetic forms. In a damped system, some energy is dissipated at each cycle of vibration and must be replaced from an external source if a steady vibration is to be maintained. Although a single physical structure may store both kinetic and potential energy, and may dissipate energy, this chapter considers only *lumped parameter systems* composed of ideal springs, masses, and dampers wherein each element has only a single function. In translational motion, displacements are defined as linear distances; in rotational motion, displacements are defined as angular motions.

TRANSLATIONAL MOTION

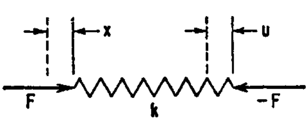

FIGURE 2.1 Linear spring.

Spring. In the linear spring shown in Fig. 2.1, the change in the length of the spring is proportional to the force acting along its length:

$$F = k(x - u) \tag{2.1}$$

The ideal spring is considered to have no mass; thus, the force acting on one end is equal and opposite to the force acting on the other end. The constant of proportionality k is the *spring constant or stiffness.*

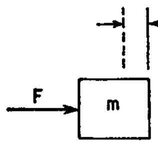

FIGURE 2.2 Rigid mass.

Mass. A mass is a rigid body (Fig. 2.2) whose acceleration $\ddot{x}$ according to Newton's second law is proportional to the resultant F of all forces acting on the mass:*

$$F = m\ddot{x} \tag{2.2}$$

Damper. In the viscous damper shown in Fig. 2.3, the applied force is proportional to the relative velocity of its connection points:

$$F = c(\dot{x} - \dot{u}) \tag{2.3}$$

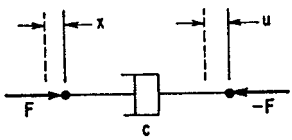

FIGURE 2.3 Viscous damper.

The constant c is the *damping coefficient,* the characteristic parameter of the damper. The ideal damper is considered to have no mass; thus the force at one end is equal and opposite to the force at the other end. *Structural damping* is considered below and several other types of damping are considered in Chap. 30.

ROTATIONAL MOTION

The elements of a mechanical system which moves with pure rotation of the parts are wholly analogous to the elements of a system that moves with pure translation. The property of a rotational system which stores kinetic energy is inertia; stiffness and damping coefficients are defined with reference to angular displacement and angular velocity, respectively. The analogous quantities and equations are listed in Table 2.1.

TABLE 2.1 Analogous Quantities in Translational and Rotational Vibrating Systems

Translational quantity	Rotational quantity
Linear displacement x	Angular displacement α
Force F	Torque M
Spring constant k	Spring constant k_r
Damping constant c	Damping constant c_r
Mass m	Moment of inertia I
Spring law $F = k(x_1 - x_2)$	Spring law $M = k_r(\alpha_1 - \alpha_2)$
Damping law $F = c(\dot{x}_1 - \dot{x}_2)$	Damping law $M = c_r(\ddot{\alpha}_1 - \dot{\alpha}_2)$
Inertia law $F = m\ddot{x}$	Inertia law $M = I\ddot{\alpha}$

* It is common to use the word *mass* in a general sense to designate a rigid body. Mathematically, the mass of the rigid body is defined by m in Eq. (2.2).

Inasmuch as the mathematical equations for a rotational system can be written by analogy from the equations for a translational system, only the latter are discussed in detail. Whenever translational systems are discussed, it is understood that corresponding equations apply to the analogous rotational system, as indicated in Table 2.1.

SINGLE DEGREE-OF-FREEDOM SYSTEM

The simplest possible vibratory system is shown in Fig. 2.4; it consists of a mass m attached by means of a spring k to an immovable support. The mass is constrained to translational motion in the direction of the X axis so that its change of position from an initial reference is described fully by the value of a single quantity x. For this reason it is called a *single degree-of-freedom system.* If the mass m is displaced from its equilibrium position and then allowed to vibrate free from further external forces, it is said to have *free vibration.* The vibration also may be forced; i.e., a continuing force acts upon the mass or the foundation experiences a continuing motion. Free and forced vibration are discussed below.

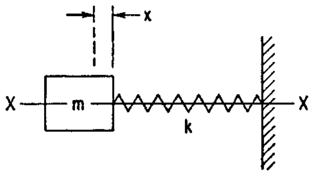

FIGURE 2.4 Undamped single degree-of-freedom system.

FREE VIBRATION WITHOUT DAMPING

Considering first the free vibration of the undamped system of Fig. 2.4, Newton's equation is written for the mass m. The force $m\ddot{x}$ exerted by the mass on the spring is equal and opposite to the force kx applied by the spring on the mass:

$$m\ddot{x} + kx = 0 \tag{2.4}$$

where $x = 0$ defines the equilibrium position of the mass.

The solution of Eq. (2.4) is

$$x = A \sin \sqrt{\frac{k}{m}}\ t + B \cos \sqrt{\frac{k}{m}}\ t \tag{2.5}$$

where the term $\sqrt{k/m}$ is the *angular natural frequency* defined by

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{rad/sec} \tag{2.6}$$

The sinusoidal oscillation of the mass repeats continuously, and the time interval to complete one cycle is the *period:*

$$\tau = \frac{2\pi}{\omega_n} \tag{2.7}$$

The reciprocal of the period is the *natural frequency:*

$$f_n = \frac{1}{\tau} = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{kg}{W}} \tag{2.8}$$

where $W = mg$ is the weight of the rigid body forming the mass of the system shown in Fig. 2.4. The relations of Eq. (2.8) are shown by the solid lines in Fig. 2.5.

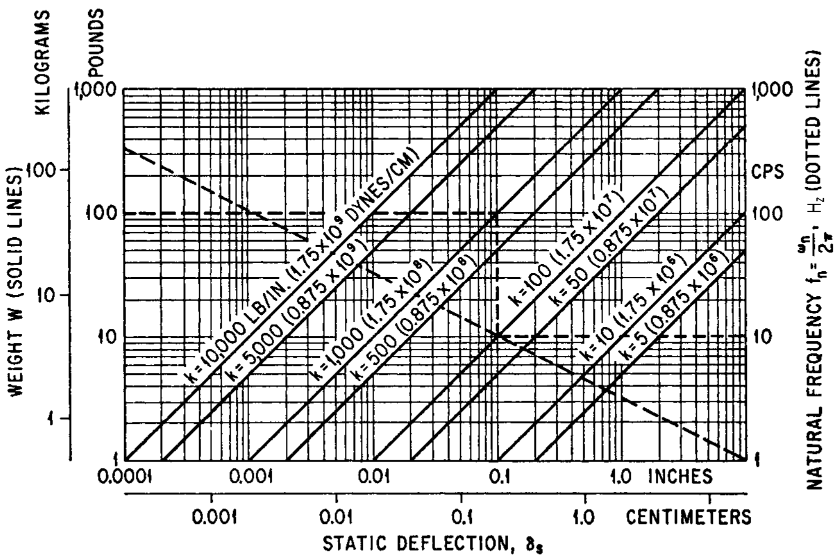

FIGURE 2.5 Natural frequency relations for a single degree-of-freedom system. Relation of natural frequency to weight of supported body and stiffness of spring [Eq. (2.8)] is shown by solid lines. Relation of natural frequency to static deflection [Eq. (2.10)] is shown by diagonal-dashed line. Example: To find natural frequency of system with $W = 100$ lb and $k = 1000$ lb/in., enter at $W = 100$ on left ordinate scale; follow the dashed line horizontally to solid line $k = 1000$, then vertically down to diagonal-dashed line, and finally horizontally to read $f_n = 10$ Hz from right ordinate scale.

Initial Conditions. In Eq. (2.5), B is the value of x at time $t = 0$, and the value of A is equal to $\dot{x}/\omega_n$ at time $t = 0$. Thus, the conditions of displacement and velocity which exist at zero time determine the subsequent oscillation completely.

Phase Angle. Equation (2.5) for the displacement in oscillatory motion can be written, introducing the frequency relation of Eq. (2.6),

$$x = A \sin \omega_n t + B \cos \omega_n t = C \sin (\omega_n t + \theta) \tag{2.9}$$

where $C = (A^2 + B^2)^{1/2}$ and $\theta = \tan^{-1} (B/A)$. The angle θ is called the *phase angle.*

Static Deflection. The static deflection of a simple mass-spring system is the deflection of spring k as a result of the gravity force of the mass, $\delta_{st} = mg/k$. (For example, the system of Fig. 2.4 would be oriented with the mass m vertically above the spring k.) Substituting this relation in Eq. (2.8),

$$f_n = \frac{1}{2\pi} \sqrt{\frac{g}{\delta_{st}}} \tag{2.10}$$

The relation of Eq. (2.10) is shown by the diagonal-dashed line in Fig. 2.5. This relation applies only when the system under consideration is both linear and elastic. For example, rubber springs tend to be nonlinear or exhibit a dynamic stiffness which differs from the static stiffness; hence, Eq. (2.10) is not applicable.

FREE VIBRATION WITH VISCOUS DAMPING

Figure 2.6 shows a single degree-of-freedom system with a viscous damper. The differential equation of motion of mass *m,* corresponding to Eq. (2.4) for the undamped system, is

$$m\ddot{x} + c\dot{x} + kx = 0 \qquad (2.11)$$

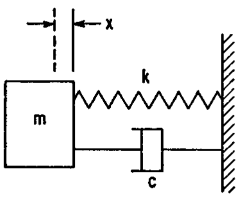

FIGURE 2.6 Single degree-of-freedom system with viscous damper.

The form of the solution of this equation depends upon whether the damping coefficient is equal to, greater than, or less than the *critical damping coefficient* c_c:

$$c_c = 2\sqrt{km} = 2m\omega_n \qquad (2.12)$$

The ratio $\zeta = c/c_c$ is defined as the *fraction of critical damping.*

Less-Than-Critical Damping. If the damping of the system is less than critical, $\zeta < 1$; then the solution of Eq. (2.11) is

$$\begin{aligned} x &= e^{-ct/2m}(A \sin \omega_d t + B \cos \omega_d t) \\ &= Ce^{-ct/2m} \sin (\omega_d t + \theta) \end{aligned} \qquad (2.13)$$

where C and θ are defined with reference to Eq. (2.9). The *damped natural frequency* is related to the undamped natural frequency of Eq. (2.6) by the equation

$$\omega_d = \omega_n(1 - \zeta^2)^{1/2} \qquad \text{rad/sec} \qquad (2.14)$$

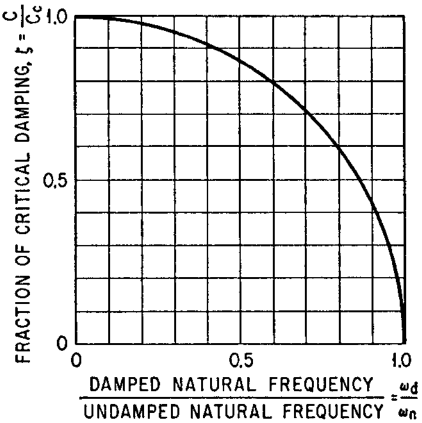

FIGURE 2.7 Damped natural frequency as a function of undamped natural frequency and fraction of critical damping.

Equation (2.14), relating the damped and undamped natural frequencies, is plotted in Fig. 2.7.

Critical Damping. When $c = c_c$, there is no oscillation and the solution of Eq. (2.11) is

$$x = (A + Bt)e^{-ct/2m} \qquad (2.15)$$

Greater-Than-Critical Damping. When $\zeta > 1$, the solution of Eq. (2.11) is

$$x = e^{-ct/2m}(Ae^{\omega_n\sqrt{\zeta^2 - 1}\, t} + Be^{-\omega_n\sqrt{\zeta^2 - 1}\, t}) \qquad (2.16)$$

This is a nonoscillatory motion; if the system is displaced from its equilibrium position, it tends to return gradually.

Logarithmic Decrement. The degree of damping in a system having $\zeta < 1$ may be defined in terms of successive peak values in a record of a free oscillation. Substituting the expression for critical damping from Eq. (2.12), the expression for free vibration of a damped system, Eq. (2.13), becomes

$$x = Ce^{-\zeta\omega_n t} \sin (\omega_d t + \theta) \tag{2.17}$$

Consider any two maxima (i.e., value of x when $dx/dt = 0$) separated by n cycles of oscillation, as shown in Fig. 2.8. Then the ratio of these maxima is

$$\frac{x_n}{x_0} = e^{-2\pi n\zeta/(1 - \zeta^2)^{1/2}} \tag{2.18}$$

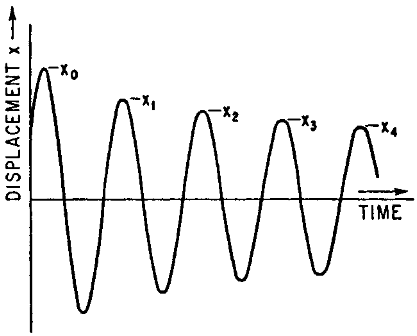

FIGURE 2.8 Trace of damped free vibration showing amplitudes of displacement maxima.

Values of x_n/x_0 are plotted in Fig. 2.9 for several values of n over the range of ζ from 0.001 to 0.10.

The *logarithmic decrement* Δ is the natural logarithm of the ratio of the amplitudes of two successive cycles of the damped free vibration:

$$\Delta = \ln \frac{x_1}{x_2} \quad \text{or} \quad \frac{x_2}{x_1} = e^{-\Delta} \tag{2.19}$$

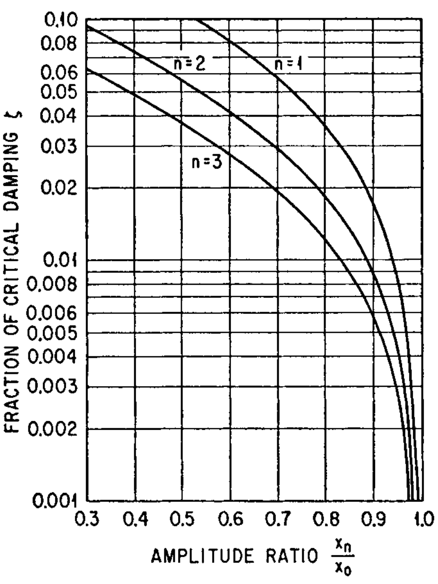

FIGURE 2.9 Effect of damping upon the ratio of displacement maxima of a damped free vibration.

[See also Eq. (37.10).] A comparison of this relation with Eq. (2.18) when $n = 1$ gives the following expression for Δ:

$$\Delta = \frac{2\pi\zeta}{(1 - \zeta^2)^{1/2}} \tag{2.20}$$

The logarithmic decrement can be expressed in terms of the difference of successive amplitudes by writing Eq. (2.19) as follows:

$$\frac{x_1 - x_2}{x_1} = 1 - \frac{x_2}{x_1} = 1 - e^{-\Delta}$$

Writing $e^{-\Delta}$ in terms of its infinite series, the following expression is obtained which gives a good approximation for $\Delta < 0.2$:

$$\frac{x_1 - x_2}{x_1} = \Delta \tag{2.21}$$

For small values of ζ (less than about 0.10), an approximate relation between the fraction of critical damping and the logarithmic decrement, from Eq. (2.20), is

$$\Delta \simeq 2\pi\zeta \tag{2.22}$$

FORCED VIBRATION

Forced vibration in this chapter refers to the motion of the system which occurs in response to a continuing excitation whose magnitude varies sinusoidally with time. (See Chaps. 8 and 23 for a treatment of the response of a simple system to step, pulse, and transient vibration excitations.) The excitation may be, alternatively, force applied to the system (generally, the force is applied to the mass of a single degree-of-freedom system) or motion of the foundation that supports the system. The resulting response of the system can be expressed in different ways, depending upon the nature of the excitation and the use to be made of the result:

1. If the excitation is a force applied to the mass of the system shown in Fig. 2.4, the result may be expressed in terms of (*a*) the amplitude of the resulting motion of the mass or (*b*) the fraction of the applied force amplitude that is transmitted through the system to the support. The former is termed the *motion response* and the latter is termed the *force transmissibility.*
2. If the excitation is a motion of the foundation, the resulting response usually is expressed in terms of the amplitude of the motion of the mass relative to the amplitude of the motion of the foundation. This is termed the *motion transmissibility* for the system.

In general, the response and transmissibility relations are functions of the forcing frequency and vary with different types and degrees of damping. Results are presented in this chapter for undamped systems and for systems with either viscous or structural damping. Corresponding results are given in Chap. 30 for systems with Coulomb damping, and for systems with either viscous or Coulomb damping in series with a linear spring.

FORCED VIBRATION WITHOUT DAMPING

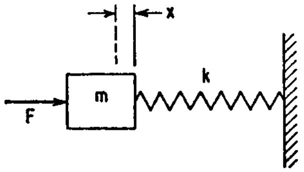

FIGURE 2.10 Undamped single degree-of-freedom system excited in forced vibration by force acting on mass.

Force Applied to Mass. When the sinusoidal force $F = F_0 \sin \omega t$ is applied to the mass of the undamped single degree-of-freedom system shown in Fig. 2.10, the differential equation of motion is

$$m\ddot{x} + kx = F_0 \sin \omega t \tag{2.23}$$

The solution of this equation is

$$x = A \sin \omega_n t + B \cos \omega_n t + \frac{F_0/k}{1 - \omega^2/\omega_n^2} \sin \omega t \tag{2.24}$$

where $\omega_n = \sqrt{k/m}$. The first two terms represent an oscillation at the undamped natural frequency ω_n. The coefficient B is the value of x at time $t = 0$, and the coefficient A may be found from the velocity at time $t = 0$. Differentiating Eq. (2.24) and setting $t = 0$,

$$\dot{x}(0) = A\omega_n + \frac{\omega F_0/k}{1 - \omega^2/\omega_n^2} \tag{2.25}$$

The value of A is found from Eq. (2.25).

The oscillation at the natural frequency ω_n gradually decays to zero in physical systems because of damping. The steady-state oscillation at forcing frequency ω is

$$x = \frac{F_0/k}{1 - \omega^2/\omega_n^2} \sin \omega t \tag{2.26}$$

This oscillation exists after a condition of equilibrium has been established by decay of the oscillation at the natural frequency ω_n and persists as long as the force F is applied.

The force transmitted to the foundation is directly proportional to the spring deflection: $F_t = kx$. Substituting x from Eq. (2.26) and defining transmissibility $T = F_t/F$,

$$T = \frac{1}{1 - \omega^2/\omega_n^2} \tag{2.27}$$

If the mass is initially at rest in the equilibrium position of the system (i.e., $x = 0$ and $\dot{x} = 0$) at time $t = 0$, the ensuing motion at time $t > 0$ is

$$x = \frac{F_0/k}{1 - \omega^2/\omega_n^2} \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t\right) \tag{2.28}$$

For large values of time, the second term disappears because of the damping inherent in any physical system, and Eq. (2.28) becomes identical to Eq. (2.26).

When the forcing frequency coincides with the natural frequency, $\omega = \omega_n$ and a condition of resonance exists. Then Eq. (2.28) is indeterminate and the expression for x may be written as

$$x = -\frac{F_0\omega}{2k} t \cos \omega t + \frac{F_0}{2k} \sin \omega t \tag{2.29}$$

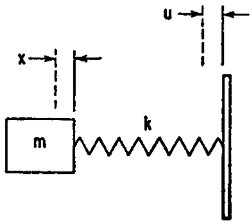

FIGURE 2.11 Undamped single degree-of-freedom system excited in forced vibration by motion of foundation.

According to Eq. (2.29), the amplitude x increases continuously with time, reaching an infinitely great value only after an infinitely great time.

Motion of Foundation. The differential equation of motion for the system of Fig. 2.11 excited by a continuing motion $u = u_0 \sin \omega t$ of the foundation is

$$m\ddot{x} = -k(x - u_0 \sin \omega t)$$

The solution of this equation is

$$x = A_1 \sin \omega_n t + B_2 \cos \omega_n t + \frac{u_0}{1 - \omega^2/\omega_n^{\ 2}} \sin \omega t$$

where $\omega_n = k/m$ and the coefficients A_1, B_1 are determined by the velocity and displacement of the mass, respectively, at time $t = 0$. The terms representing oscillation at the natural frequency are damped out ultimately, and the ratio of amplitudes is defined in terms of transmissibility T:

$$\frac{x_0}{u_0} = T = \frac{1}{1 - \omega^2/\omega_n^{\ 2}} \tag{2.30}$$

where $x = x_0 \sin \omega t$. Thus, in the forced vibration of an undamped single degree-of-freedom system, the motion response, the force transmissibility, and the motion transmissibility are numerically equal.

FORCED VIBRATION WITH VISCOUS DAMPING

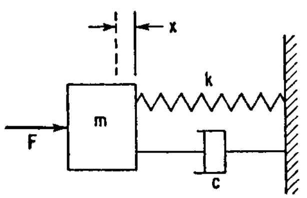

FIGURE 2.12 Single degree-of-freedom system with viscous damping, excited in forced vibration by force acting on mass.

Force Applied to Mass. The differential equation of motion for the single degree-of-freedom system with viscous damping shown in Fig. 2.12, when the excitation is a force $F = F_0 \sin \omega t$ applied to the mass, is

$$m\ddot{x} + c\dot{x} + kx = F_0 \sin \omega t \tag{2.31}$$

Equation (2.31) corresponds to Eq. (2.23) for forced vibration of an undamped system; its solution would correspond to Eq. (2.24) in that it includes terms representing oscillation at the natural frequency. In a damped system, however, these terms are damped out rapidly and only the steady-state solution usually is considered. The resulting motion occurs at the forcing frequency ω; when the damping coefficient c is greater than zero, the phase between the force and resulting motion is different than zero. Thus, the response may be written

$$x = R \sin (\omega t - \theta) = A_1 \sin \omega t + B_1 \cos \omega t \tag{2.32}$$

Substituting this relation in Eq. (2.31), the following result is obtained:

$$\frac{x}{F_0/k} = \frac{\sin(\omega t - \theta)}{\sqrt{(1 - \omega^2/\omega_n{}^2)^2 + (2\zeta\omega/\omega_n)^2}} = R_d \sin(\omega t - \theta) \tag{2.33}$$

where $$\theta = \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1 - \omega^2/\omega_n{}^2}\right)$$

and R_d is a *dimensionless response factor* giving the ratio of the amplitude of the vibratory displacement to the spring displacement that would occur if the force F were applied statically. At very low frequencies R_d is approximately equal to 1; it rises to a peak near ω_n and approaches zero as ω becomes very large. The displacement response is defined at these frequency conditions as follows:

$$x \simeq \left(\frac{F_0}{k}\right) \sin \omega t \qquad [\omega << \omega_n]$$

$$x = \frac{F_0}{2k\zeta} \sin\left(\omega_n t + \frac{\pi}{2}\right) = -\frac{F_0 \cos \omega_n t}{c\omega_n} \qquad [\omega = \omega_n] \tag{2.34}$$

$$x \simeq \frac{\omega_n{}^2 F_0}{\omega^2 k} \sin(\omega t + \pi) = \frac{F_0}{m\omega^2} \sin \omega t \qquad [\omega >> \omega_n]$$

For the above three frequency conditions, the vibrating system is sometimes described as *spring-controlled, damper-controlled,* and *mass-controlled,* respectively, depending on which element is primarily responsible for the system behavior.

Curves showing the dimensionless response factor R_d as a function of the frequency ratio ω/ω_n are plotted in Fig. 2.13 on the coordinate lines having a positive 45° slope. Curves of the phase angle θ are plotted in Fig. 2.14. A phase angle between 180 and 360° cannot exist in this case since this would mean that the damper is furnishing energy to the system rather than dissipating it.

An alternative form of Eqs. (2.33) and (2.34) is

$$\frac{x}{F_0/k} = \frac{(1 - \omega^2/\omega n^2) \sin \omega t - 2\zeta(\omega/\omega_n) \cos \omega t}{(1 - \omega^2/\omega_n{}^2)^2 + (2\zeta\omega/\omega_n)^2}$$

$$= (R_d)_x \sin \omega t + (R_d)_R \cos \omega t \tag{2.35}$$

This shows the components of the response which are in phase $[(R_d)_x \sin \omega t]$ and 90° out of phase $[(R_d)_R \cos \omega t]$ with the force. Curves of $(R_d)_x$ and $(R_d)_R$ are plotted as a function of the frequency ratio ω/ω_n in Figs. 2.15 and 2.16.

Velocity and Acceleration Response. The shape of the response curves changes distinctly if velocity $\dot{x}$ or acceleration $\ddot{x}$ is plotted instead of displacement x. Differentiating Eq. (2.33),

$$\frac{\dot{x}}{F_0/\sqrt{km}} = \frac{\omega}{\omega_n} R_d \cos(\omega t - \theta) = R_v \cos(\omega t - \theta) \tag{2.36}$$

The acceleration response is obtained by differentiating Eq. (2.36):

$$\frac{\ddot{x}}{F_0/m} = -\frac{\omega^2}{\omega_n{}^2} R_d \sin(\omega t - \theta) = -R_a \sin(\omega t - \theta) \tag{2.37}$$

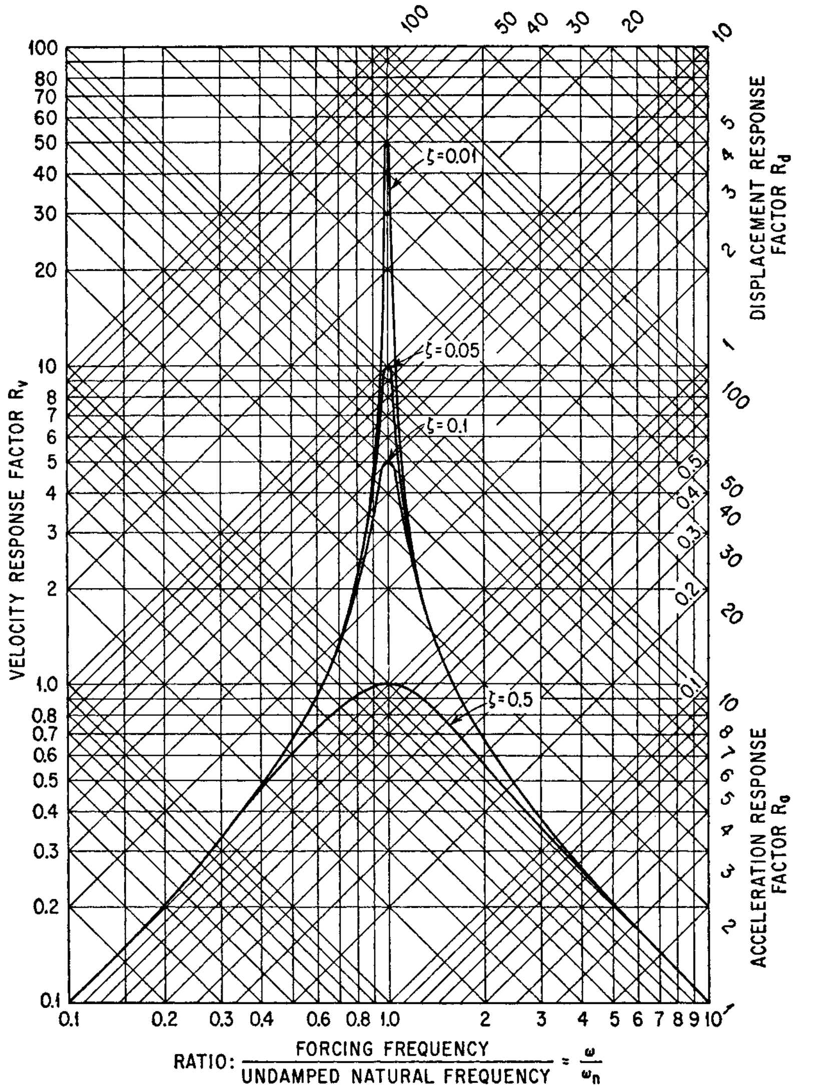

FIGURE 2.13 Response factors for a viscous-damped single degree-of-freedom system excited in forced vibration by a force acting on the mass. The velocity response factor shown by horizontal lines is defined by Eq. (2.36); the displacement response factor shown by diagonal lines of positive slope is defined by Eq. (2.33); and the acceleration response factor shown by diagonal lines of negative slope is defined by Eq. (2.37).

The velocity and acceleration response factors defined by Eqs. (2.36) and (2.37) are shown graphically in Fig. 2.13, the former to the horizontal coordinates and the latter to the coordinates having a negative 45° slope. Note that the velocity response factor approaches zero as $\omega \to 0$ and $\omega \to \infty$, whereas the acceleration response factor approaches 0 as $\omega \to 0$ and approaches unity as $\omega \to \infty$.

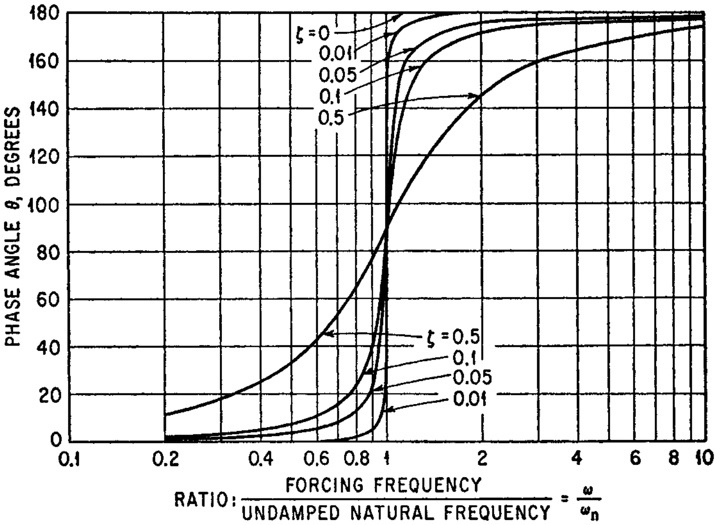

FIGURE 2.14 Phase angle between the response displacement and the excitation force for a single degree-of-freedom system with viscous damping, excited by a force acting on the mass of the system.

Force Transmission. The force transmitted to the foundation of the system is

$$F_T = c\dot{x} + kx \tag{2.38}$$

Since the forces $c\dot{x}$ and kx are 90° out of phase, the magnitude of the transmitted force is

$$|F_T| = \sqrt{c^2\dot{x}^2 + k^2x^2} \tag{2.39}$$

The ratio of the transmitted force F_T to the applied force F_0 can be expressed in terms of transmissibility T:

$$\frac{F_T}{F_0} = T \sin(\omega t - \psi) \tag{2.40}$$

where

$$T = \sqrt{\frac{1 + (2\zeta\omega/\omega_n)^2}{(1 - \omega^2/\omega_n^2)^2 + (2\zeta\omega/\omega_n)^2}} \tag{2.41}$$

and

$$\psi = \tan^{-1} \frac{2\zeta(\omega/\omega_n)^3}{1 - \omega^2/\omega_n^2 + 4\zeta^2\omega^2/\omega_n^2}$$

The transmissibility T and phase angle ψ are shown in Figs. 2.17 and 2.18, respectively, as a function of the frequency ratio ω/ω_n and for several values of the fraction of critical damping ζ.

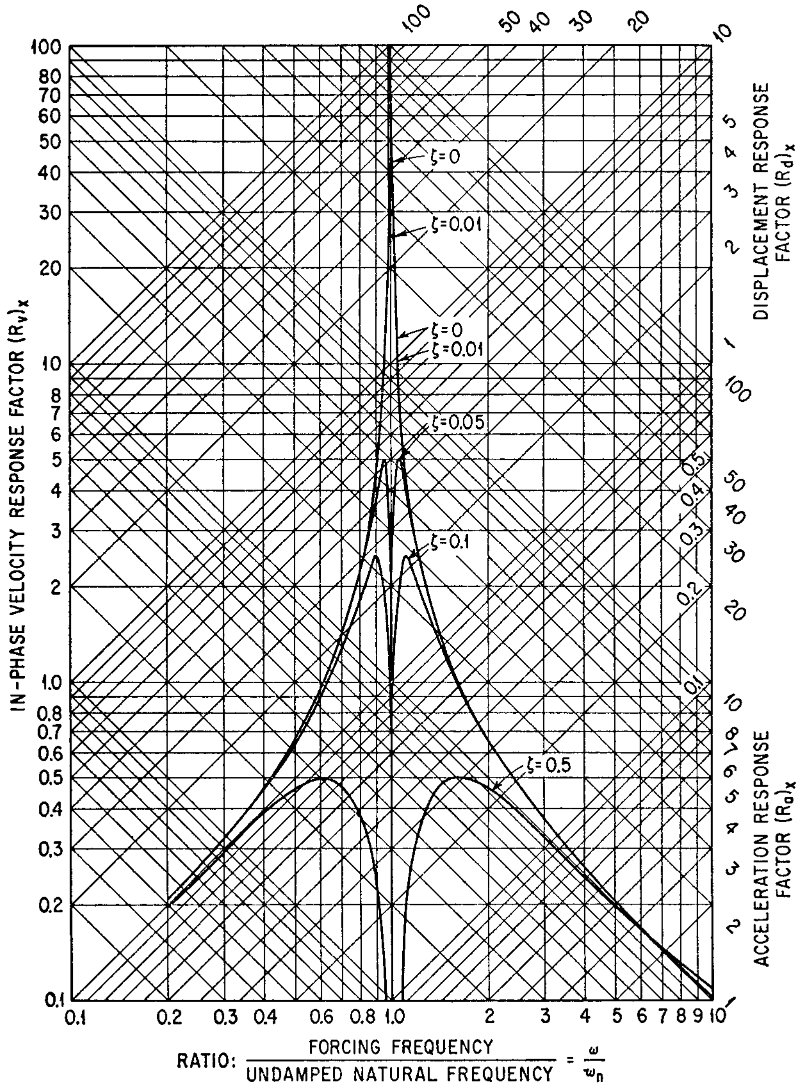

FIGURE 2.15 In-phase component of response factor of a viscous-damped system in forced vibration. All values of the response factor for $\omega/\omega_n > 1$ are negative but are plotted without regard for sign. The fraction of critical damping is denoted by ζ.

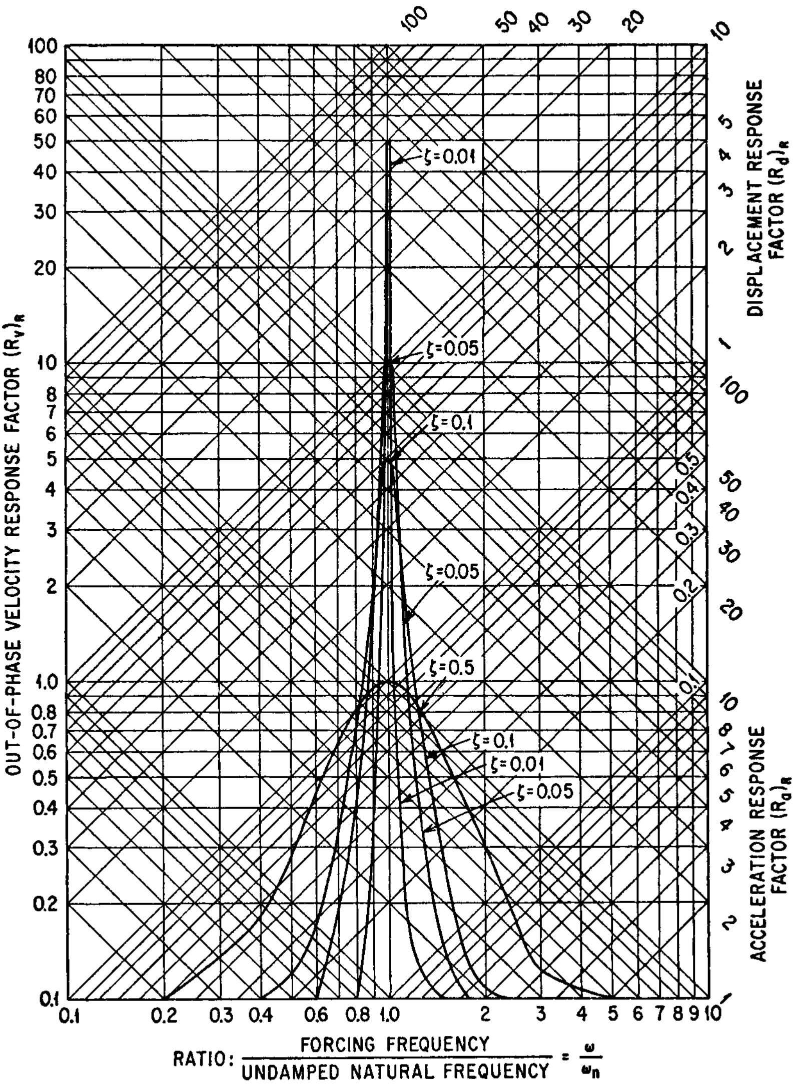

FIGURE 2.16 Out-of-phase component of response factor of a viscous-damped system in forced vibration. The fraction of critical damping is denoted by ζ.

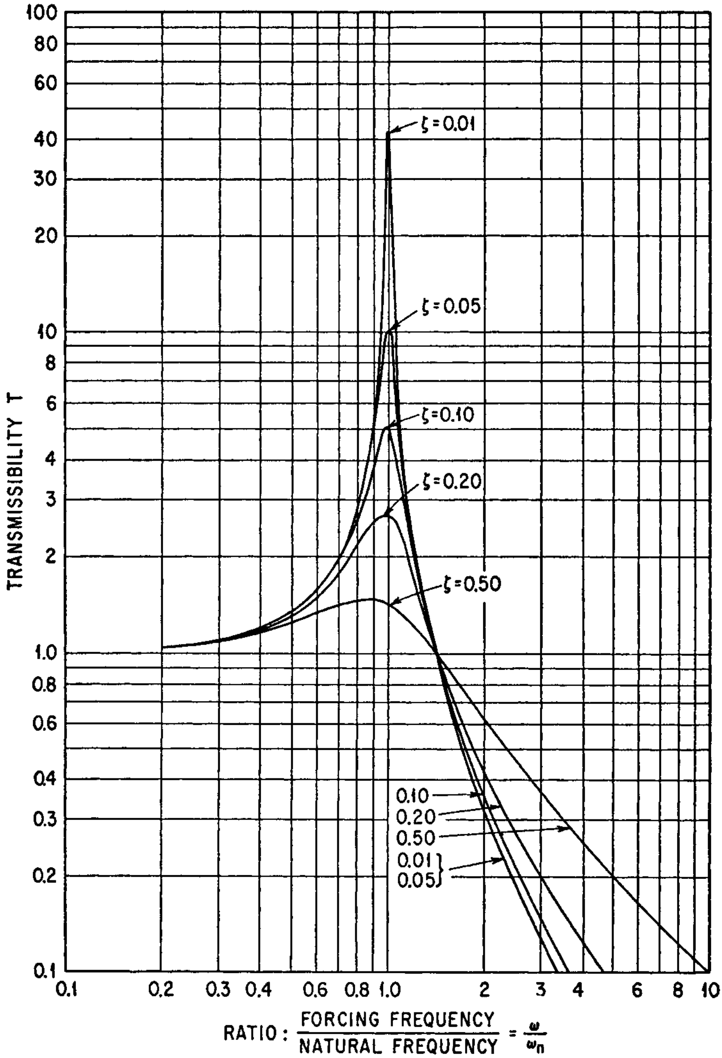

FIGURE 2.17 Transmissibility of a viscous-damped system. Force transmissibility and motion transmissibility are identical numerically. The fraction of critical damping is denoted by ζ.

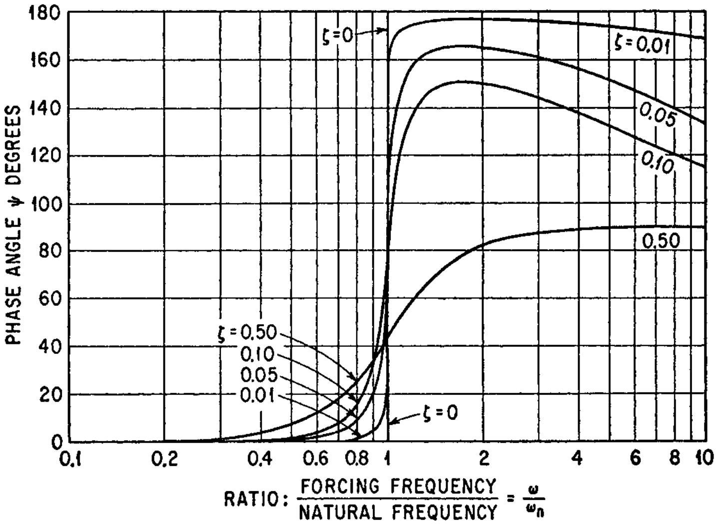

FIGURE 2.18 Phase angle of force transmission (or motion transmission) of a viscous-damped system excited (1) by force acting on mass and (2) by motion of foundation. The fraction of critical damping is denoted by ζ.

Hysteresis. When the viscous damped, single degree-of-freedom system shown in Fig. 2.12 undergoes vibration defined by

$$x = x_0 \sin \omega t \tag{2.42}$$

the net force exerted on the mass by the spring and damper is

$$F = kx_0 \sin \omega t + c\omega x_0 \cos \omega t \tag{2.43}$$

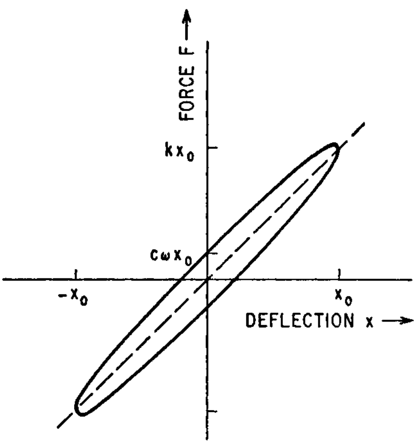

FIGURE 2.19 Hysteresis curve for a spring and viscous damper in parallel.

Equations (2.42) and (2.43) define the relation between F and x; this relation is the ellipse shown in Fig. 2.19. The energy dissipated in one cycle of oscillation is

$$W = \int_T^{T + 2\pi/\omega} F \frac{dx}{dt} dt = \pi c \omega x_0^2 \tag{2.44}$$

Motion of Foundation. The excitation for the elastic system shown in Fig. 2.20 may be a motion $u(t)$ of the foundation. The differential equation of motion for the system is

$$m\ddot{x} + c(\dot{x} - \dot{u}) + k(x - u) = 0 \tag{2.45}$$

Consider the motion of the foundation to be a displacement that varies sinu-

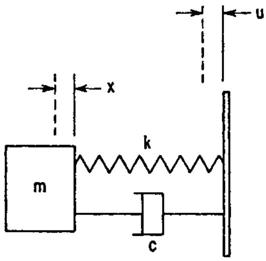

FIGURE 2.20 Single degree-of-freedom system with viscous damper, excited in forced vibration by foundation motion.

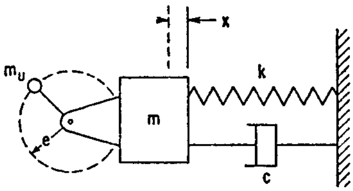

FIGURE 2.21 Single degree-of-freedom system with viscous damper, excited in forced vibration by rotating eccentric weight.

soidally with time, $u = u_0 \sin \omega t$. A steady-state condition exists after the oscillations at the natural frequency ω_n are damped out, defined by the displacement x of mass m:

$$x = Tu_0 \sin (\omega t - \psi) \tag{2.46}$$

where T and ψ are defined in connection with Eq. (2.40) and are shown graphically in Figs. 2.17 and 2.18, respectively. Thus, the motion transmissibility T in Eq. (2.46) is identical numerically to the force transmissibility T in Eq. (2.40). The motion of the foundation and of the mass m may be expressed in any consistent units, such as displacement, velocity, or acceleration, and the same expression for T applies in each case.

Vibration Due to a Rotating Eccentric Weight. In the mass-spring-damper system shown in Fig. 2.21, a mass m_u is mounted by a shaft and bearings to the mass m. The mass m_u follows a circular path of radius e with respect to the bearings. The component of displacement in the X direction of m_u relative to m is

$$x_3 - x_1 = e \sin \omega t \tag{2.47}$$

where x_3 and x_1 are the absolute displacements of m_u and m, respectively, in the X direction; e is the length of the arm supporting the mass m_u; and ω is the angular velocity of the arm in radians per second. The differential equation of motion for the system is

$$m\ddot{x}_1 + m_u\ddot{x}_3 + c\dot{x}_1 + kx_1 = 0 \tag{2.48}$$

Differentiating Eq. (2.47) with respect to time, solving for $\ddot{x}_3$, and substituting in Eq. (2.48):

$$(m + m_u)\ddot{x}_1 + c\dot{x}_1 + kx_1 = m_u e\omega^2 \sin \omega t \tag{2.49}$$

Equation (2.49) is of the same form as Eq. (2.31); thus, the response relations of Eqs. (2.33), (2.36), and (2.37) apply by substituting $(m + m_u)$ for m and $m_u e\omega^2$ for F_0. The resulting displacement, velocity, and acceleration responses are

$$\frac{x_1}{m_u e\omega^2} = R_d \sin (\omega t - \theta) \qquad \frac{\dot{x}_1\sqrt{km}}{m_u e\omega^2} = R_v \cos (\omega t - \theta)$$
$$\frac{\ddot{x}_1 m}{m_u e\omega^2} = - R_a \sin (\omega t - \theta) \tag{2.50}$$

Resonance Frequencies. The peak values of the displacement, velocity, and acceleration response of a system undergoing forced, steady-state vibration occur at slightly different forcing frequencies. Since a *resonance frequency* is defined as the frequency for which the response is a maximum, a simple system has three resonance frequencies if defined only generally. The natural frequency is different from any of the resonance frequencies. The relations among the several resonance frequencies, the damped natural frequency, and the undamped natural frequency ω_n are:

Displacement resonance frequency: $\omega_n(1 - 2\zeta^2)^{1/2}$
Velocity resonance frequency: ω_n
Acceleration resonance frequency: $\omega_n/(1 - 2\zeta^2)^{1/2}$
Damped natural frequency: $\omega_n(1 - \zeta^2)^{1/2}$

For the degree of damping usually embodied in physical systems, the difference among the three resonance frequencies is negligible.

Resonance, Bandwidth, and the Quality Factor *Q*. Damping in a system can be determined by noting the maximum response, i.e., the response at the resonance frequency as indicated by the maximum value of R_v in Eq. (2.36). This is defined by the factor Q sometimes used in electrical engineering terminology and defined with respect to mechanical vibration as

$$Q = (R_v)_{\max} = 1/2\zeta$$

The maximum acceleration and displacement responses are slightly larger, being

$$(R_d)_{\max} = (R_a)_{\max} = \frac{(R_v)_{\max}}{(1 - \zeta^2)^{1/2}}$$

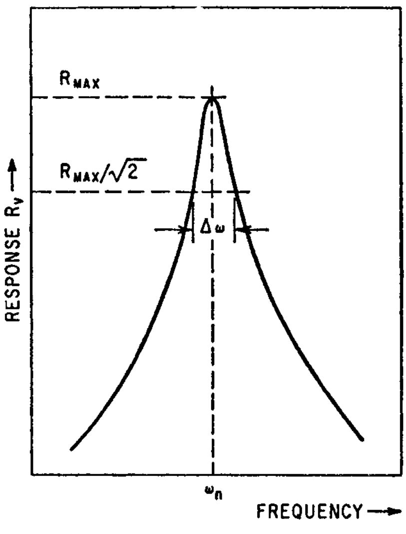

FIGURE 2.22 Response curve showing bandwidth at "half-power point."

The damping in a system is also indicated by the sharpness or width of the response curve in the vicinity of a resonance frequency ω_n. Designating the width as a frequency increment $\Delta\omega$ measured at the "half-power point" (i.e., at a value of R equal to $R_{\max}/2$), as illustrated in Fig. 2.22, the damping of the system is defined to a good approximation by

$$\frac{\Delta\omega}{\omega_n} = \frac{1}{Q} = 2\zeta \qquad (2.51)$$

for values of ζ less than 0.1. The quantity $\Delta\omega$, known as the *bandwidth,* is commonly represented by the letter B.

Structural Damping. The energy dissipated by the damper is known as *hysteresis loss;* as indicated by Eq. (2.44), it is proportional to the forcing frequency ω. However, the hysteresis loss of many engineering structures has been found to be independent of frequency. To provide a better model for defining the *structural damping* experienced during vibration, an arbitrary damping term $k\mathfrak{g} = c\omega$ is introduced. In effect, this defines the damping force as being equal to the viscous damping force at some frequency, depending upon the value of $\mathfrak{g}$, but being invariant with frequency. The relation of the damping force F to the displacement x is defined by an ellipse similar to Fig. 2.19, and the displacement response of the system is described by an expression corresponding to Eq. (2.33) as follows:

$$\frac{x}{F_0/k} = R_g \sin(\omega t - \theta) = \frac{\sin(\omega t - \theta)}{\sqrt{(1 - \omega^2/\omega_n^{\,2})^2 + \mathfrak{g}^2}} \tag{2.52}$$

where $\mathfrak{g} = 2\zeta\omega/\omega_n$. The resonance frequency is ω_n, and the value of R_g at resonance is $1/\mathfrak{g} = Q$.

The equations for the hysteresis ellipse for structural damping are

$$\begin{aligned} F &= kx_0 (\sin \omega t + \mathfrak{g} \cos \omega t) \\ x &= x_0 \sin \omega t \end{aligned} \tag{2.53}$$

UNDAMPED MULTIPLE DEGREE-OF-FREEDOM SYSTEMS

An elastic system sometimes cannot be described adequately by a model having only one mass but rather must be represented by a system of two or more masses considered to be *point masses* or *particles* having no rotational inertia. If a group of particles is bound together by essentially rigid connections, it behaves as a rigid body having both mass (significant for translational motion) and moment of inertia (significant for rotational motion). There is no limit to the number of masses that may be used to represent a system. For example, each mass in a model representing a beam may be an infinitely thin slice representing a cross section of the beam; a differential equation is required to treat this continuous distribution of mass.

DEGREES-OF-FREEDOM

The number of independent parameters required to define the distance of all the masses from their reference positions is called the number of *degrees-of-freedom N*. For example, if there are N masses in a system constrained to move only in translation in the X and Y directions, the system has $2N$ degrees-of-freedom. A continuous system such as a beam has an infinitely large number of degrees-of-freedom.

For each degree-of-freedom (each coordinate of motion of each mass) a differential equation can be written in one of the following alternative forms:

$$m_j \ddot{x}_j = F_{xj} \qquad I_k \ddot{\alpha}_k = M_{\alpha k} \tag{2.54}$$

where F_{xj} is the component in the X direction of all external, spring, and damper forces acting on the mass having the jth degree-of-freedom, and $M_{\alpha k}$ is the component about the α axis of all torques acting on the body having the kth degree-of-freedom. The moment of inertia of the mass about the α axis is designated by I_k. (This is assumed for the present analysis to be a principal axis of inertia, and product of inertia terms are neglected. See Chap. 3 for a more detailed discussion.) Equations (2.54) are identical in form and can be represented by

$$m_j\ddot{x}_j = F_j \tag{2.55}$$

where F_j is the resultant of all forces (or torques) acting on the system in the jth degree-of-freedom, $\ddot{x}_j$ is the acceleration (translational or rotational) of the system in the jth degree-of-freedom, and m_j is the mass (or moment of inertia) in the jth degree-of-freedom. Thus, the terms defining the motion of the system (displacement, velocity, and acceleration) and the deflections of structures may be either translational or rotational, depending upon the type of coordinate. Similarly, the "force" acting on a system may be either a force or a torque, depending upon the type of coordinate. For example, if a system has n bodies each free to move in three translational modes and three rotational modes, there would be $6n$ equations of the form of Eq. (2.55), one for each degree-of-freedom.

DEFINING A SYSTEM AND ITS EXCITATION

The first step in analyzing any physical structure is to represent it by a mathematical model which will have essentially the same dynamic behavior. A suitable number and distribution of masses, springs, and dampers must be chosen, and the input forces or foundation motions must be defined. The model should have sufficient degrees-of-freedom to determine the modes which will have significant response to the exciting force or motion.

The properties of a system that must be known are the natural frequencies ω_n, the normal mode shapes D_{jn}, the damping of the respective modes, and the mass distribution m_j. The detailed distributions of stiffness and damping of a system are not used directly but rather appear indirectly as the properties of the respective modes. The characteristic properties of the modes may be determined experimentally as well as analytically.

STIFFNESS COEFFICIENTS

The spring system of a structure of N degrees-of-freedom can be defined completely by a set of N^2 *stiffness coefficients.* A stiffness coefficient K_{jk} is the change in spring force acting on the jth degree-of-freedom when only the kth degree-of-freedom is slowly displaced a unit amount in the negative direction. This definition is a generalization of the linear, elastic spring defined by Eq. (2.1). Stiffness coefficients have the characteristic of reciprocity, i.e., $K_{jk} = K_{kj}$. The number of independent stiffness coefficients is $(N^2 + N)/2$.

The total elastic force acting on the jth degree-of-freedom is the sum of the effects of the displacements in all of the degrees-of-freedom:

$$F_{el} = -\sum_{k=1}^{N} K_{jk}x_k \tag{2.56}$$

Inserting the spring force F_{el} from Eq. (2.56) in Eq. (2.55) together with the external forces F_j results in the n equations:

$$m_j\ddot{x}_j = F_j - \sum_{k} K_{jk}x_k \tag{2.56a}$$

FREE VIBRATION

When the external forces are zero, the preceding equations become

$$m_j\ddot{x}_j + \sum_k K_{jk}x_k = 0 \tag{2.57}$$

Solutions of Eq. (2.57) have the form

$$x_j = D_j \sin(\omega t + \theta) \tag{2.58}$$

Substituting Eq. (2.58) in Eq. (2.57),

$$m_j\omega^2 D_j = \sum_k K_{jk}D_k \tag{2.59}$$

This is a set of n linear algebraic equations with n unknown values of D. A solution of these equations for values of D other than zero can be obtained only if the determinant of the coefficients of the D's is zero:

$$\begin{vmatrix} (m_1\omega^2 - K_{11}) & -K_{12} & \cdot & \cdot & -K_{in} \\ -K_{21} & (m_2\omega^2 - K_{22}) & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -K_{ni} & \cdot & \cdot & \cdot & (m_n\omega^2 - K_{nn}) \end{vmatrix} = 0 \tag{2.60}$$

Equation (2.60) is an algebraic equation of the nth degree in ω^2; it is called the *frequency equation* since it defines n values of ω which satisfy Eq. (2.57). The roots are all real; some may be equal, and others may be zero. These values of frequency determined from Eq. (2.60) are the frequencies at which the system can oscillate in the absence of external forces. These frequencies are the *natural frequencies* ω_n of the system. Depending upon the initial conditions under which vibration of the system is initiated, the oscillations may occur at any or all of the natural frequencies and at any amplitude.

Example 2.1. Consider the three degree-of-freedom system shown in Fig. 2.23; it consists of three equal masses m and a foundation connected in series by three

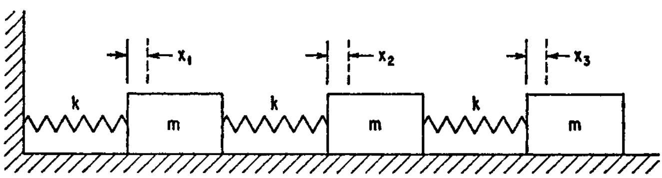

FIGURE 2.23 Undamped three degree-of-freedom system on foundation.

equal springs k. The absolute displacements of the masses are x_1, x_2, and x_3. The stiffness coefficients (see section entitled *Stiffness Coefficients*) are thus $K_{11} = 2k$,

$K_{22} = 2k$, $K_{33} = k$, $K_{12} = K_{21} = -k$, $K_{23} = K_{32} = -k$, and $K_{13} = K_{31} = 0$. The frequency equation is given by the determinant, Eq. (2.60),

$$\begin{vmatrix} (m\omega^2 - 2k) & k & 0 \\ k & (m\omega^2 - 2k) & k \\ 0 & k & (m\omega^2 - k) \end{vmatrix} = 0$$

The determinant expands to the following polynomial:

$$\left(\frac{m\omega^2}{k}\right)^3 - 5\left(\frac{m\omega^2}{k}\right)^2 + 6\left(\frac{m\omega^2}{k}\right) - 1 = 0$$

Solving for ω,

$$\omega = 0.445\sqrt{\frac{k}{m}}, \qquad 1.25\sqrt{\frac{k}{m}}, \qquad 1.80\sqrt{\frac{k}{m}}$$

Normal Modes of Vibration. A structure vibrating at only one of its natural frequencies ω_n does so with a characteristic pattern of amplitude distribution called a *normal mode of vibration.* A normal mode is defined by a set of values of D_{jn} [see Eq. (2.58)] which satisfy Eq. (2.59) when $\omega = \omega_n$:

$$\omega_n^2 m_j D_{jn} = \sum_k K_{jn} D_{kn} \tag{2.61}$$

A set of values of D_{jn} which form a normal mode is independent of the absolute values of D_{jn} but depends only on their relative values. To define a mode shape by a unique set of numbers, any arbitrary *normalizing condition* which is desired can be used. A condition often used is to set $D_{1n} = 1$ but $\sum_j m_j D_{jn}^2 = 1$ and $\sum_j m_j D_{jn}^2 = \sum_j m_j$ also may be found convenient.

Orthogonality of Normal Modes. The usefulness of normal modes in dealing with multiple degree-of-freedom systems is due largely to the orthogonality of the normal modes. It can be shown that the set of inertia forces $\omega_n^2 m_j D_{jn}$ for one mode does not work on the set of deflections D_{jm} of another mode of the structure:

$$\sum_j m_j D_{jm} D_{jn} = 0 \qquad [m \neq n] \tag{2.62}$$

This is the *orthogonality condition.*

Normal Modes and Generalized Coordinates. Any set of N deflections x_j can be expressed as the sum of normal mode amplitudes:

$$x_j = \sum_{n=1}^{N} q_n D_{jn} \tag{2.63}$$

The numerical values of the D_{jn}'s are fixed by some normalizing condition, and a set of values of the N variables q_n can be found to match any set of x_j's. The N values of q_n constitute a set of *generalized coordinates* which can be used to define the position coordinates x_j of all parts of the structure. The q's are also known as the amplitudes of the normal modes, and are functions of time. Equation (2.63) may be differentiated to obtain

$$\ddot{x}_j = \sum_{n=1}^{N} \ddot{q}_n D_{jn} \tag{2.64}$$

Any quantity which is distributed over the j coordinates can be represented by a linear transformation similar to Eq. (2.63). It is convenient now to introduce the parameter γ_n relating D_{jn} and F_j/m_j as follows:

$$\frac{F_j}{m_j} = \sum_n \gamma_n D_{jn} \tag{2.65}$$

where F_j may be zero for certain values of n.

FORCED MOTION

Substituting the expressions in generalized coordinates, Eqs. (2.63) to (2.65), in the basic equation of motion, Eq. (2.56*a*),

$$m_j \sum_n \ddot{q}_n D_{jn} + \sum_k k_{jk} \sum_n q_n D_{kn} - m_j \sum_n \gamma_n D_{jn} = 0 \tag{2.66}$$

The center term in Eq. (2.66) may be simplified by applying Eq. (2.61) and the equation rewritten as follows:

$$\sum_n (\ddot{q}_n + \omega_n^2 q_n - \gamma_n) m_j D_{jn} = 0 \tag{2.67}$$

Multiplying Eqs. (2.67) by D_{jm} and taking the sum over j (i.e., adding all the equations together),

$$\sum_n (\ddot{q}_n + \omega_n^2 q_n - \gamma_n) \sum_j m_j D_{jn} D_{jm} = 0$$

All terms of the sum over n are zero, except for the term for which $m = n$, according to the orthogonality condition of Eq. (2.62). Then since $\sum_j m_j D_{jn}^2$ is not zero, it follows that

$$\ddot{q}_n + \omega_n^2 q_n - \gamma_n = 0$$

for every value of n from 1 to N.

An expression for γ_n may be found by using the orthogonality condition again. Multiplying Eq. (2.65) by $m_j D_{jm}$ and taking the sum taken over j,

$$\sum_j F_j D_{jm} = \sum_n \gamma_n \sum_j m_j D_{jn} D_{jm} \tag{2.68}$$

All the terms of the sum over n are zero except when $n = m$, according to Eq. (2.62), and Eq. (2.68) reduces to

$$\gamma_n = \frac{\sum_j F_j D_{jn}}{\sum_j m_j D_{jn}^2} \tag{2.69}$$

Then the differential equation for the response of any generalized coordinate to the externally applied forces F_j is

$$\ddot{q}_n + \omega_n^2 q_n = \gamma_n = \frac{\sum_j F_j D_{jn}}{\sum_j m_j D_{jn}^2} \tag{2.70}$$

where $\Sigma F_j D_{jn}$ is the generalized force, i.e., the total work done by all external forces during a small displacement δq_n divided by δq_n, and $\Sigma m_j D_{jn}^2$ is the generalized mass.

Thus the amplitude q_n of each normal mode is governed by its own equation, independent of the other normal modes, and responds as a simple mass-spring system. Equation (2.70) is a generalized form of Eq. (2.23).

The forces F_j may be any functions of time. Any equation for the response of an undamped mass-spring system applies to each mode of a complex structure by substituting:

$$\begin{aligned}
&\text{The } \textit{generalized coordinate } q_n \text{ for } x \\
&\text{The } \textit{generalized force } \sum_j F_j D_{jn} \text{ for } F \\
&\text{The } \textit{generalized mass } \sum_j m_j D_{jn} \text{ for } m \\
&\text{The } \textit{mode natural frequency } \omega_n \text{ for } \omega_n
\end{aligned} \tag{2.71}$$

Response to Sinusoidal Forces. If a system is subjected to one or more sinusoidal forces $F_j = F_{0j} \sin \omega t$, the response is found from Eq. (2.26) by noting that $k = m\omega_n^2$ [Eq. (2.6)] and then substituting from Eq. (2.71):

$$q_n = \frac{\sum_j F_{0j} D_{jn}}{\omega_n^2 \sum_j m_j D_{jn}^2} \, \frac{\sin \omega t}{(1 - \omega^2/\omega_n^2)} \tag{2.72}$$

Then the displacement of the kth degree-of-freedom, from Eq. (2.63), is

$$x_k = \sum_{n=1}^{N} \frac{D_{kn} \sum_j F_{0j} D_{jn} \sin \omega t}{\omega_n^2 \sum_j m_j D_{jn}^2 (1 - \omega^2/\omega_n^2)} \tag{2.73}$$

This is the general equation for the response to sinusoidal forces of an undamped system of N degrees-of-freedom. The application of the equation to systems free in space or attached to immovable foundations is discussed below.

Example 2.2. Consider the system shown in Fig. 2.24; it consists of three equal masses m connected in series by two equal springs k. The system is free in space and

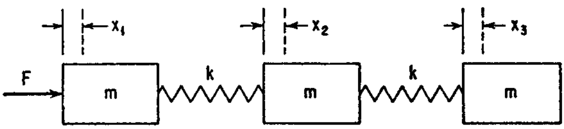

FIGURE 2.24 Undamped three degree-of-freedom system acted on by sinusoidal force.

a force $F \sin \omega t$ acts on the first mass. Absolute displacements of the masses are x_1, x_2, and x_3. Determine the acceleration $\ddot{x}_3$. The stiffness coefficients (see section entitled *Stiffness Coefficients*) are $K_{11} = K_{33} = k$, $K_{22} = 2k$, $K_{12} = K_{21} = -k$, $K_{13} = K_{31} = 0$, and $K_{23} = K_{32} = -k$. Substituting in Eq. (2.60), the frequency equation is

$$\begin{vmatrix} (m\omega^2 - k) & k & 0 \\ k & (m\omega^2 - 2k) & k \\ 0 & k & (m\omega^2 - k) \end{vmatrix} = 0$$

The roots are $\omega_1 = 0$, $\omega_2 = \sqrt{k/m}$, and $\omega_3 = \sqrt{3k/m}$. The zero value for one of the natural frequencies indicates that the entire system translates without deflection of the springs. The mode shapes are now determined by substituting from Eq. (2.58) in Eq. (2.57), noting that $\ddot{x} = -D\omega^2$, and writing Eq. (2.59) for each of the three masses in each of the oscillatory modes 2 and 3:

$$mD_{21}\left(\frac{k}{m}\right) = K_{11}D_{21} + K_{21}D_{22} + K_{31}D_{23}$$

$$mD_{22}\left(\frac{k}{m}\right) = K_{12}D_{21} + K_{22}D_{22} + K_{32}D_{23}$$

$$mD_{23}\left(\frac{k}{m}\right) = K_{13}D_{21} + K_{23}D_{22} + K_{33}D_{23}$$

$$mD_{31}\left(\frac{3k}{m}\right) = K_{11}D_{31} + K_{21}D_{32} + K_{31}D_{33}$$

$$mD_{32}\left(\frac{3k}{m}\right) = K_{12}D_{31} + K_{22}D_{32} + K_{32}D_{33}$$

$$mD_{33}\left(\frac{3k}{m}\right) = K_{13}D_{31} + K_{23}D_{32} + K_{33}D_{33}$$

where the first subscript on the D's indicates the mode number (according to ω_1 and ω_2 above) and the second subscript indicates the displacement amplitude of the particular mass. The values of the stiffness coefficients K are calculated above. The mode shapes are defined by the relative displacements of the masses. Thus, assigning values of unit displacement to the first mass (i.e., $D_{21} = D_{31} = 1$), the above equations may be solved simultaneously for the D's:

$$D_{21} = 1 \qquad D_{22} = 0 \qquad D_{23} = -1$$

$$D_{31} = 1 \qquad D_{32} = -2 \qquad D_{33} = 1$$

Substituting these values of D in Eq. (2.71), the generalized masses are determined: $M_2 = 2m$, $M_3 = 6m$.

Equation (2.73) then can be used to write the expression for acceleration $\ddot{x}_3$:

$$\ddot{x}_3 = \left[\frac{1}{3m} + \frac{(\omega^2/\omega_2^{\,2})(-1)(+1)}{2m(1 - \omega^2/\omega_2^{\,2})} + \frac{(\omega^2/\omega_3^{\,2})(+1)(+1)}{6m(1 - \omega^2/\omega_3^{\,2})}\right] F_1 \sin \omega t$$

Free and Fixed Systems. For a structure which is free in space, there are six "normal modes" corresponding to $\omega_n = 0$. These represent motion of the structure without relative motion of its parts; this is rigid body motion with six degrees-of-freedom.

The rigid body modes all may be described by equations of the form

$$D_{jm} = a_{jm} D_m \qquad [m = 1,2,\ldots,6]$$

where D_m is a motion of the rigid body in the m coordinate and a is the displacement of the jth degree-of-freedom when D_m is moved a unit amount. The geometry of the structure determines the nature of a_{jm}. For example, if D_m is a rotation about the Z axis, $a_{jm} = 0$ for all modes of motion in which j represents rotation about the X or Y axis and $a_{jm} = 0$ if j represents translation parallel to the Z axis. If D_{jm} is a translational mode of motion parallel to X or Y, it is necessary that a_{jm} be proportional to the distance r_j of m_j from the Z axis and to the sine of the angle between r_j and the jth direction. The above relations may be applied to an elastic body. Such a body moves as a rigid body in the gross sense in that all particles of the body move together generally but may experience relative vibratory motion. The orthogonality condition applied to the relation between any rigid body mode D_{jm} and any oscillatory mode D_{jn} yields

$$\sum_j m_j D_{jn} D_{jm} = \sum_j m_j a_{jm} D_{jn} = 0 \qquad \begin{bmatrix} m \le 6 \\ n > 6 \end{bmatrix} \tag{2.74}$$

These relations are used in computations of oscillatory modes, and show that normal modes of vibration involve no net translation or rotation of a body.

A system attached to a fixed foundation may be considered as a system free in space in which one or more "foundation" masses or moments of inertia are infinite. Motion of the system as a rigid body is determined entirely by the motion of the foundation. The amplitude of an oscillatory mode representing motion of the foundation is zero; i.e., $M_j D_{jn}^2 = 0$ for the infinite mass. However, Eq. (2.73) applies equally well regardless of the size of the masses.

Foundation Motion. If a system is small relative to its foundation, it may be assumed to have no effect on the motion of the foundation. Consider a foundation of large but unknown mass m_0 having a motion $x_0 \sin \omega t$, the consequence of some unknown force

$$F_0 \sin \omega t = -m_0 x_0 \omega^2 \sin \omega t \tag{2.75}$$

acting on m_0 in the x_0 direction. Equation (2.73) is applicable to this case upon substituting

$$-m_0 x_0 \omega^2 D_{0n} = \sum_j F_{0j} D_{jn} \tag{2.76}$$

where D_{0n} is the amplitude of the foundation (the 0 degree-of-freedom) in the nth mode.

The oscillatory modes of the system are subject to Eqs. (2.74):

$$\sum_j = 0\; m_j a_{jm} D_{jn} = 0$$

Separating the 0th degree-of-freedom from the other degrees-of-freedom:

$$\sum_{j=0} m_j a_{jm} D_{jn} = m_0 a_{0m} D_{0n} + \sum_{j=1} m_j\, a_{jm} D_{jn}$$

If m_0 approaches infinity as a limit, D_{0n} approaches zero and motion of the system as a rigid body is identical with the motion of the foundation. Thus, a_{0m} approaches unity for motion in which $m = 0$, and approaches zero for motion in which $m \neq 0$. In the limit:

$$\lim_{m_0 \to \infty} m_0 D_{0n} = -\sum_j m_j a_{j0} D_{jn} \tag{2.77}$$

Substituting this result in Eq. (2.76),

$$\lim_{m_0 \to \infty} \sum_j F_{0j} D_{jn} = x_0 \omega^2 \sum_j m_j a_{j0} D_{jn} \tag{2.78}$$

The generalized mass in Eq. (2.73) includes the term $m_0 D_{0n}{}^2$, but this becomes zero as m_0 becomes infinite.

The equation for response of a system to motion of its foundation is obtained by substituting Eq. (2.78) in Eq. (2.73):

$$x_k = \sum_{n=1}^{N} \frac{\omega^2}{\omega_n{}^2} D_{kn} \frac{\sum_j m_j a_{j0} D_{jn} x_0 \sin \omega t}{\sum_j m_j D_{jn}{}^2 (1 - \omega^2/\omega_n{}^2)} + x_0 \sin \omega t \tag{2.79}$$

DAMPED MULTIPLE DEGREE-OF-FREEDOM SYSTEMS

Consider a set of masses interconnected by a network of springs and acted upon by external forces, with a network of dampers acting in parallel with the springs. The viscous dampers produce forces on the masses which are determined in a manner analogous to that used to determine spring forces and summarized by Eq. (2.56). The damping force acting on the jth degree-of-freedom is

$$(F_d)_j = -\sum_k C_{jk} \dot{x}_k \tag{2.80}$$

where C_{jk} is the resultant force on the jth degree-of-freedom due to a unit velocity of the kth degree-of-freedom.

In general, the distribution of damper sizes in a system need not be related to the spring or mass sizes. Thus, the dampers may couple the normal modes together, allowing motion of one mode to affect that of another. Then the equations of response are not easily separable into independent normal mode equations. However, there are two types of damping distribution which do not couple the normal modes. These are known as *uniform viscous damping* and *uniform mass damping*.

UNIFORM VISCOUS DAMPING

Uniform damping is an appropriate model for systems in which the damping effect is an inherent property of the spring material. Each spring is considered to have a damper acting in parallel with it, and the ratio of damping coefficient to stiffness coefficient is the same for each spring of the system. Thus, for all values of j and k,

$$\frac{C_{jk}}{k_{jk}} = 2\mathcal{G} \tag{2.81}$$

where $\mathcal{G}$ is a constant.

Substituting from Eq. (2.81) in Eq. (2.80),

$$-(F_d)_j = \sum_k C_{jk}\dot{x}_k = 2\mathcal{G} \sum_k k_{jk}\dot{x}_k \tag{2.82}$$

Since the damping forces are "external" forces with respect to the mass-spring system, the forces $(F_d)_j$ can be added to the external forces in Eq. (2.70) to form the equation of motion:

$$\ddot{q}_n + \omega_n{}^2 q_n = \frac{\sum_j (F_d)_j D_{jn} + \sum_j F_j D_{jn}}{\sum_j m_j D_{jn}{}^2} \tag{2.83}$$

Combining Eqs. (2.61), (2.63), and (2.82), the summation involving $(F_d)_j$ in Eq. (2.83) may be written as follows:

$$\sum_j (F_d)_j D_{jn} = -2\mathcal{G}\omega_n{}^2 \dot{q}_n \sum_j m_j D_{jn}{}^2 \tag{2.84}$$

Substituting Eq. (2.84) in Eq. (2.83),

$$\ddot{q}_n + 2\mathcal{G}\omega_n{}^2 \dot{q}_n + \omega_n{}^2 q_n = \gamma_n \tag{2.85}$$

Comparison of Eq. (2.85) with Eq. (2.31) shows that each mode of the system responds as a simple damped oscillator.

The damping term $2\mathcal{G}\omega_n{}^2$ in Eq. (2.85) corresponds to $2\zeta\omega_n$ in Eq. (2.31) for a simple system. Thus, $\mathcal{G}\omega_n$ may be considered the critical damping ratio of each mode. Note that the effective damping for a particular mode varies directly as the natural frequency of the mode.

Free Vibration. If a system with uniform viscous damping is disturbed from its equilibrium position and released at time $t = 0$ to vibrate freely, the applicable equation of motion is obtained from Eq. (2.85) by substituting $2\zeta\omega$ for $2\mathcal{G}\omega_n{}^2$ and letting $\gamma_n = 0$:

$$\ddot{q}_n + 2\zeta\omega_n \dot{q}_n + \omega_n{}^2 q_n = 0 \tag{2.86}$$

The solution of Eq. (2.86) for less than critical damping is

$$x_j(t) = \sum_n D_{jn} e^{-\zeta\omega_n t}(A_n \sin \omega_d t + B_n \cos \omega_d t) \tag{2.87}$$

where $\omega_d = \omega_n(1 - \zeta^2)^{1/2}$.

The values of A and B are determined by the displacement $x_j(0)$ and velocity $\dot{x}_j(0)$ at time $t = 0$:

$$x_j(0) = \sum_n B_n D_{jn}$$

$$\dot{x}_j(0) = \sum_n (A_n \omega_{dn} - B_n \zeta\omega_n) D_{jn}$$

Applying the orthogonality relation of Eq. (2.62) in the manner used to derive Eq. (2.69),

$$B_n = \frac{\sum_j x_j(0) m_j D_{jn}}{\sum_j m_j D_{jn}^2}$$
$$A_n \omega_{dn} - B_n \zeta \omega_{dn} = \frac{\sum_j \dot{x}_j(0) m_j D_{jn}}{\sum_j m_j D_{jn}^2} \tag{2.88}$$

Thus each mode undergoes a decaying oscillation at the damped natural frequency for the particular mode, and the amplitude of each mode decays from its initial value, which is determined by the initial displacements and velocities.

UNIFORM STRUCTURAL DAMPING

To avoid the dependence of viscous damping upon frequency, as indicated by Eq. (2.85), the uniform viscous damping factor $\mathcal{G}$ is replaced by $\mathfrak{g}/\omega$ for uniform structural damping. This corresponds to the structural damping parameter $\mathfrak{g}$ in Eqs. (2.52) and (2.53) for sinusoidal vibration of a simple system. Thus, Eq. (2.85) for the response of a mode to a sinusoidal force of frequency ω is

$$\ddot{q}_n + \frac{2\mathfrak{g}}{\omega} \omega_n^2 \dot{q}_n + \omega_n^2 q_n = \gamma_n \tag{2.89}$$

The amplification factor at resonance ($Q = 1/\mathfrak{g}$) has the same value in all modes.

UNIFORM MASS DAMPING

If the damping force on each mass is proportional to the magnitude of the mass,

$$(F_d)_j = -B m_j \dot{x}_j \tag{2.90}$$

where B is a constant. For example, Eq. (2.90) would apply to a uniform beam immersed in a viscous fluid.

Substituting as $\dot{x}_j$ in Eq. (2.90) the derivative of Eq. (2.63),

$$\Sigma (F_d)_j D_{jn} = -B \sum_j m_j D_{jn} \sum_m \dot{q}_m D_{jm} \tag{2.91}$$

Because of the orthogonality condition, Eq. (2.62):

$$\Sigma (F_d)_j D_{jn} = -B \dot{q}_n \sum_j m_j D_{jn}^2$$

Substituting from Eq. (2.91) in Eq. (2.83), the differential equation for the system is

$$\ddot{q}_n + B \dot{q}_n + \omega_n^2 q_n = \gamma_n \tag{2.92}$$

where the damping term B corresponds to $2\zeta\omega$ for a simple oscillator, Eq. (2.31). Then $B/2\omega_n$ represents the fraction of critical damping for each mode, a quantity which diminishes with increasing frequency.

GENERAL EQUATION FOR FORCED VIBRATION

All the equations for response of a linear system to a sinusoidal excitation may be regarded as special cases of the following general equation:

$$x_k = \sum_{n=1}^{N} \frac{D_{kn}}{\omega_n^2} \frac{F_n}{m_n} R_n \sin(\omega t - \theta_n) \tag{2.93}$$

where x_k = displacement of structure in kth degree-of-freedom
N = number of degrees-of-freedom, including those of the foundation
D_{kn} = amplitude of kth degree-of-freedom in nth normal mode
F_n = generalized force for nth mode
m_n = generalized mass for nth mode
R_n = response factor, a function of the frequency ratio ω/ω_n (Fig. 2.13)
θ_n = phase angle (Fig. 2.14)

Equation (2.93) is of sufficient generality to cover a wide variety of cases, including excitation by external forces or foundation motion, viscous or structural damping, rotational and translational degrees-of-freedom, and from one to an infinite number of degrees-of-freedom.

LAGRANGIAN EQUATIONS

The differential equations of motion for a vibrating system sometimes are derived more conveniently in terms of kinetic and potential energies of the system than by the application of Newton's laws of motion in a form requiring the determination of the forces acting on each mass of the system. The formulation of the equations in terms of the energies, known as Lagrangian equations, is expressed as follows:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_n} - \frac{\partial T}{\partial q_n} + \frac{\partial V}{\partial q_n} = F_n \tag{2.94}$$

where T = total kinetic energy of system
V = total potential energy of system
q_n = generalized coordinate—a displacement
$\dot{q}_n$ = velocity at generalized coordinate q_n
F_n = generalized force, the portion of the total forces not related to the potential energy of the system (gravity and spring forces appear in the potential energy expressions and are not included here)

The method of applying Eq. (2.94) is to select a number of independent coordinates (generalized coordinates) equal to the number of degrees-of-freedom, and to write expressions for total kinetic energy T and total potential energy V. Differentiation of these expressions successively with respect to each of the chosen coordinates leads to a number of equations similar to Eq. (2.94), one for each coordinate (degree-of-freedom). These are the applicable differential equations and may be solved by any suitable method.

Example 2.3. Consider free vibration of the three degree-of-freedom system shown in Fig. 2.23; it consists of three equal masses m connected in tandem by equal springs k. Take as coordinates the three absolute displacements x_1, x_2, and x_3. The kinetic energy of the system is

$$T = \tfrac{1}{2}m(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$$

The potential energy of the system is

$$V = \frac{k}{2}\,[x_1^2 + (x_1 - x_2)^2 + (x_2 - x_3)^2] = \frac{k}{2}\,(2x_1^2 + 2x_2^2 + x_3^2 - 2x_1x_2 - 2x_2x_3)$$

Differentiating the expression for the kinetic energy successively with respect to the velocities,

$$\frac{\partial T}{\partial \dot{x}_1} = m\dot{x}_1 \qquad \frac{\partial T}{\partial \dot{x}_2} = m\dot{x}_2 \qquad \frac{\partial T}{\partial \dot{x}_3} = m\dot{x}_3$$

The kinetic energy is not a function of displacement; therefore, the second term in Eq. (2.94) is zero. The partial derivatives with respect to the displacement coordinates are

$$\frac{\partial V}{\partial x_1} = 2kx_1 - kx_2 \qquad \frac{\partial V}{\partial x_2} = 2kx_2 - kx_1 - kx_3 \qquad \frac{\partial V}{\partial x_3} = kx_3 - kx_2$$

In free vibration, the generalized force term in Eq. (2.93) is zero. Then, substituting the derivatives of the kinetic and potential energies from above into Eq. (2.94),

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 + 2kx_2 - kx_1 - kx_3 = 0$$

$$m\ddot{x}_3 + kx_3 - kx_2 = 0$$

The natural frequencies of the system may be determined by placing the preceding set of simultaneous equations in determinant form, in accordance with Eq. (2.60):

$$\begin{vmatrix} (m\omega^2 - 2k) & k & 0 \\ k & (m\omega^2 - 2k) & k \\ 0 & k & (m\omega^2 - k) \end{vmatrix} = 0$$

The natural frequencies are equal to the values of ω that satisfy the preceding determinant equation.

Example 2.4. Consider the compound pendulum of mass m shown in Fig. 2.25, having its center-of-gravity located a distance l from the axis of rotation. The moment of inertia is I about an axis through the center-of-gravity. The position of the mass is defined by three coordinates, x and y to define the location of the center-of-gravity and θ to define the angle of rotation.

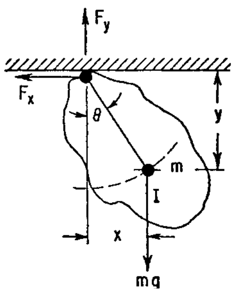

FIGURE 2.25 Forces and motions of a compound pendulum.

The *equations of constraint* are $y = l\cos\theta$; $x = l\sin\theta$. Each equation of constraint reduces the number of degrees-of-freedom by 1; thus the pendulum is a one degree-of-freedom system whose position is defined uniquely by θ alone.

The kinetic energy of the pendulum is

$$T = \tfrac{1}{2}(I + ml^2)\dot{\theta}^2$$

The potential energy is

$$V = mgl(1 - \cos\theta)$$

Then

$$\frac{\partial T}{\partial \dot{\theta}} = (I + ml^2)\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = (I + ml^2)\ddot{\theta}$$

$$\frac{\partial T}{\partial \theta} = 0 \qquad \frac{\partial V}{\partial \theta} = mgl\sin\theta$$

Substituting these expressions in Eq. (2.94), the differential equation for the pendulum is

$$(I + ml^2)\ddot{\theta} + mgl\sin\theta = 0$$

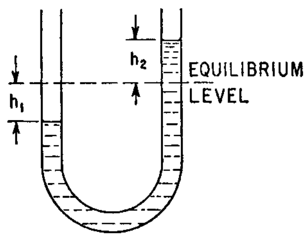

FIGURE 2.26 Water column in a U-tube.

Example 2.5. Consider oscillation of the water in the U-tube shown in Fig. 2.26. If the displacements of the water levels in the arms of a uniform-diameter U-tube are h_1 and h_2, then conservation of matter requires that $h_1 = -h_2$. The kinetic energy of the water flowing in the tube with velocity h_1 is

$$T = \tfrac{1}{2}\rho S l \dot{h}_1^2$$

where ρ is the water density, S is the cross-section area of the tube, and l is the developed length of the water column. The potential energy (difference in potential energy between arms of tube) is

$$V = S\rho g h_1^2$$

Taking h_1 as the generalized coordinate, differentiating the expressions for energy, and substituting in Eq. (2.94),

$$S\rho l\ddot{h}_1 + 2\rho g S h_1 = 0$$

Dividing through by ρSl,

$$\ddot{h}_1 + \frac{2g}{l}h_1 = 0$$

This is the differential equation for a simple oscillating system of natural frequency ω_n, where

$$\omega_n = \sqrt{\frac{2g}{l}}$$